MATHEMATICS

B.Sc, Part-II, Paper-III

Topic: Convergency and Divergency of Series

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Definitions

(i) Series: A series is a succession of quantities which are forming in order according to some definite law.

(ii) Convergent series: If the sum of the first n terms of a series tends to a finite limit S, so that the sum can, by sufficiently increasing n, be made to differ from S by less than any assigned quantity, however small, the series is said to be convergent and S is called its sum.

Thus $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ is a convergent series whose sum is 2.

(iii) Divergent series: If the sum of the first n terms of a series increases numerically without limit as n is increased indefinitely, the series is said to be divergent.

Thus $1+2+3+4\cdots$ is a divergent series.

(iv) Oscillating series: If the sum of the first n terms of a series does not increase indefinitely as n increased without limit, and yet does not approach to any determinate limit, the series is neither convergent not divergent, and is said to be oscillatory.

Thus the series $1-1+1-1+\cdots$ is an oscillatory series, for the sum of n terms is 1 or 0 according as n is odd or even.

THEOREM Prove that a necessary condition for a series $\sum u_n$ to converge is that $u_n \to 0$ as $n \to \infty$.
Show by an example that the condition is not sufficient.

Proof: Let $S_n = u_1+u_2+u_3+\cdots+u_n$.

Then $S_{n-1} = u_1+u_2+u_3+\cdots+u_{n-1}$.

Now $S_n - S_{n-1} = u_n$ ——— (1)

Since $\sum u_n$ is convergent, so let $S_n \to S$ as $n \to \infty$; then S_{n-1} will also tend to S as $n \to \infty$.

Taking limits of both sides of (1) we obtain

$$\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\,(S_n - S_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0.$$

The converse of this theorem is not necessarily true. Let us consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \text{ to } \infty.$$

Here $u_n = \frac{1}{n}$ which tends to zero as $n \to \infty$.

But $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ to ∞ is not convergent.

Thus we find that though $u_n \to 0$ as $n \to \infty$ still the series $\sum u_n$ is not convergent.

THEOREM : First Form of Comparison Test

If after a particular term, the ratio of the corresponding terms of two series of positive terms be always finite and non zone, the series will be both convergent or both divergent.

Proof. Let $\sum u_n$ and $\sum v_n$ be the two given series whose $\frac{u_n}{v_n}$ is always finite for $n \geq m$ where m is a particular integer.

Let $\frac{u_r}{v_r}$ be the greatest fraction and $\frac{u_t}{v_t}$ the least fraction.

Also let $\frac{u_r}{v_r} = k_1$ and $\frac{u_t}{v_t} = k_2$.

$\therefore \frac{u_m}{v_m}, \frac{u_{m+1}}{v_{m+1}} \ldots$ are all less than or equal to k_1

$\therefore u_n \leq k_1 v_n$ for $n \geq m$

i.e,

$$u_m \leq k_1 v_m$$
$$u_{m+1} \leq k_1 v_{m+1}$$
$$u_{m+2} \leq k_1 v_{m+2}$$
$$\cdots \quad \cdots$$
$$\cdots \quad \cdots$$
$$u_n \leq k_1 v_n$$

Adding we get

$$u_m + u_{m+1} + u_{m+2} + \cdots + u_n \leq k_1(v_m + v_{m+1} + v_{m+2} + \cdots + v_n)$$

i.e. $$\frac{u_m + u_{m+1} + \cdots + u_n}{v_m + v_{m+1} + \cdots + v_n} \leq k_1.$$

similarly $$\frac{u_m + u_{m+1} + \cdots + u_n}{v_m + v_{m+1} + \cdots + v_n} \geq k_2.$$

But k_1 and k_2 are finite

So let $\frac{u_m + u_{m+1} + \cdots + u_n}{v_m + v_{m+1} + \cdots + v_n} = L$, where L is finite which lies between k_1 and k_2.

$\therefore u_m+u_{m+1}+\cdots+u_n = L(V_m+V_{m+1}+\cdots+V_n)$ for all n.

This shows that $u_m+u_{m+1}+\cdots$ and $V_m+V_{m+1}+\cdots$ converge or diverge together. So adding a finite number of terms we find that $u_1+u_2+\cdots$ and $V_1+V_2+\cdots$ converge or diverge together.

THEOREM 3. Second Form Of Comparison Test

If $\sum u_n$ and $\sum V_n$ be two finite series of positive terms such that $\lim \frac{u_n}{V_n} \to l\,(>0)$ as $n \to \infty$, then prove that the two series are either both convergent or both divergent.

Proof Since $\frac{u_n}{V_n} \to l$, therefore a positive integer m can be obtained corresponding to a number ϵ, however small, such that

$$\left|\frac{u_n}{V_n} - l\right| < \epsilon \text{, when } n \geq m. \quad \text{i.e., } l-\epsilon < \frac{u_n}{V_n} < l+\epsilon \text{, when } n \geq m$$

since $V_n > 0$ for all n, therefore multiplying the above inequalities by V_n, we obtain

$$(l-\epsilon)V_n < u_n < (l+\epsilon)V_n \text{, when } n \geq m$$

Thus we get two inequalities:

(i) $(l-\epsilon)V_n < u_n$, where $n \geq m$

and (ii) $u_n < (l+\epsilon)V_n$, when $n \geq m$.

Let us take $(l-\epsilon)V_n < u_n$, when $n \geq m$.

If we choose $l-\epsilon = k > 0$ then

$$kV_n < u_n \text{, when } n \geq m$$

i.e., $k(V_m+V_{m+1}+V_{n+2}+\cdots+V_n) < u_m+u_{m+1}+u_{m+2}+\cdots+u_n$

i.e. $k\sum_1^n V_n < \sum_1^n u_n + k(V_1+V_2+\cdots+V_{m-1}) - (u_1+u_2+\cdots+u_{m-1})$

i.e.

$$k\sum_1^n V_n < \sum_1^n u_n + A, \qquad (1)$$

where $A = k(V_1+V_2+\cdots+V_{m-1}) - (u_1+u_2+\cdots+u_{m-1})$.

From (1) we find that if $\sum_1^n u_n$ is bounded, so is $\sum_1^n V_n$.

Consequently if $\sum u_n$ is convergent, then $\sum V_n$ is also convergent.

Again from (1) we find that if $\sum_1^n V_n$ be not bounded above, Then $\sum_{1}^{n} u_n$ is also not bounded above i.e. if $\sum v_n$ diverges, then $\sum u_n$ also diverges.

Again let us take the second inequality

$$u_n < (l+\epsilon) v_n, \text{ when } n \geq m.$$

If we choose $l+\epsilon = p > 0$, then

$$u_n < p v_n, \text{ when } n \geq m$$

i.e. $(u_m + u_{m+1} + u_{m+2} + \cdots + u_n) < p(v_m + v_{m+1} + v_{m+2} + \cdots v_n)$

i.e. $\sum_{1}^{n} u_n < p \sum_{1}^{n} v_n + (u_1 + u_2 + \cdots + u_{m-1}) - p(v_1 + v_2 + \cdots + v_{m-1})$

i.e.

$$\sum_{1}^{n} u_n < p \sum_{1}^{n} v_n + B \quad \cdots\cdots \quad (2)$$

where $B = u_1 + u_2 + \cdots + u_{m-1} - p(v_1 + v_2 + \cdots + v_{m-1})$.

From (2) we find that if $\sum_{1}^{n} v_n$ is bounded, so is $\sum_{1}^{n} u_n$.

consequently if $\sum v_n$ converges then $\sum u_n$ also converges.

Again from (2) we find that if $\sum_{1}^{n} u_n$ be not bounded above, then $\sum_{1}^{n} v_n$ is also not bounded above

i.e. if $\sum u_n$ is divergent, then $\sum v_n$ is also divergent.

Hence the two series converge or diverge together.